Some Graphs On Near Divisor Cordial-II

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Abstract: - A Near divisor cordial labeling of a graph G with vertex set V is a bijection f from V to \{1,2,\ldots, |V| - 1, |V| +1\}, such that if each edge uv is assigned the label 1 if \( f(u) \) divides \( f(v) \) (or) if \( f(v) \) divides \( f(u) \) and 0 otherwise, then the number of edges labelled with 0 and the number of edges labelled with 1 differ by almost 1. If a graph admits Near divisor cordial labeling then it is called Near divisor cordial graph. In this paper, We proved graphs such as \( J(n+1,n), S_n, B_n, m, K_1,n, S(K_1,n), K_2,n, K_3,n, <K_1,n(1), K_1,n(2)>, <K_1,n(1), K_1,n(2), K_1,n(3)>, <K_1<n(1), K_1<n(2), K_1<n(3)> \) are Near divisor cordial (NDC).

Keywords
Cordial labelling, Divisor cordial labelling and Near divisor cordial labelling.

1. INTRODUCTION

By a graph, we mean a finite undirected graphs without loops and multiple edges for terms not defined here. We refer to Harary [3]

**Definition 1.1 [1]:**
Let \( G = (V,E) \) be a graph. A mapping \( f : V(G) \rightarrow \{0,1\} \) is called binary vertex labelling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

Cahit [1] defined cordial labelling as follows

**Definition 1.2:**
A binary vertex labelling of a graph \( G \) is called a cordial labelling if

\[
|v f(0) - v f(1)| \leq 1 \quad \text{and} \quad |e f(0) - e f(1)| \leq 1.
\]

A graph \( G \) is cordial if it admits cordial labelling.

Here \( f^* : E(G) \rightarrow \{0,1\} \) is given by \( f^*(e) = |f(u) - f(v)| \). \( v f(0), v f(1) \) be the number of vertices of \( G \) having labels 0 and 1 respectively under \( f \) and \( e f(0), e f(1) \) be the number of edges of \( G \) having 0 and 1 respectively under \( f^* \).

The concept of divisor cordial labelling is introduced by R. Varatharajan, S. Navaneetha Krishnan and K. Nagarajan [5] and defined as follows:

**Definition 1.3 [5]:**
Let \( G = (V,E) \) be a simple graph and \( f : v \rightarrow \{1,2,\ldots,|V|\} \) be a bijection. For each edge \( uv \), assign the label 1 if either \( f(u) \mid f(v) \) or \( f(v) \mid f(u) \) and the label 0 otherwise. \( f \) is called divisor cordial labelling if \( |e f(0) - e f(1)| \leq 1 \). Note that \( K_7 \) is not divisor cordial but it is Near divisor cordial and \( K_{1,2m} \) is divisor cordial but it is not Near divisor cordial. Hence the above definition is meaningful.

The following definitions are useful for proving theorems.

**Definition 2.2:** For integers \( m, n \geq 0 \), we consider the graph jellyfish \( J(m,n) \) with vertex set \( V(J(m,n)) = \{u,v,x,y\} \)

\[
U \{x_1,x_2,\ldots, x_m\} \cup \{y_1,y_2, \ldots, y_n\} \quad \text{and the edge set} \quad E(J(m,n)) = \{(u,x), (u,y), (v,x), (v,y)\} \cup \{(xi,yi) / 1 \leq i \leq m\} \quad \text{U} \{(yi,yi) / 1 \leq i \leq n\}.
\]

**Definition 2.3:** The graph \( P_n + K_1 \) is called a shell.

**Definition 2.4:** The Bistar \( B_m, n \) is the graph obtained from \( K_2 \) by identifying the center vertices of \( K_1,m \) and \( K_1,n \) at the vertices of \( K_2 \) respectively. \( B_m, n \) is often denoted by \( B(m) \).

The Complete bipartite graph \( K_1,n \) is called a Star Graph and it is denoted by \( S(n) \).

The sub division of the star \( K_1,n \) is a tree obtained from the star \( K_1,n \) by adding a new pendant edge to each of the existing \( n \) pendant vertices.

**Definition 2.5:** Consider two stars \( K_1,n(1) \) and \( K_1,n(2) \). Then \( G = < K_1,n(1), K_1,n(2) > \) is the graph obtained by joining apex vertices of stars to a new vertex \( x \).

Note that \( G \) has \( 2n+3 \) vertices and \( 2n+2 \) edges.

**Definition 2.6:** Consider \( t \) copies of stars namely \( K_1,n(1), K_1,n(2), \ldots, K_1,n(t) \). Then \( G = < K_1,n(1), K_1,n(2), \ldots, K_1,n(t) > \) is the graph obtained by joining apex vertices of each \( K_1,n(m-1) \) and \( K_1,n(m) \) to a new vertex \( x=1 \) where \( 2 \leq m \leq t \).

Note that \( G \) has \( t(n+2) \) vertices and \( t(n+2)+2 \) edges.

**THEOREM 2.7:**
The graph \( J(n+1,n) \) is Near divisor cordial.

**Proof:**
Let \( V(J(n+1,n)) = \{u_1,u_2,\ldots, un,w_1,w_2, w_3, w_4, v_1,v_2,v_3,\ldots, vn+1\} \)
Example 2.8:

\[ E(J(n+1,n)) = \{ w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_2w_4 \} \cup \{ w_{3i} / 1 \leq i \leq n \} \cup \{ w_{1i} / 1 \leq i \leq n+1 \} \]

Define \( f(w_1) = 1 \) and \( f(w_3) = S \) such that \( S \) is the largest prime number such that \( S \leq 2n+6 \) and \( S \neq 2n+5 \).

Label the remaining vertices from \( \{2,3,...,S-1,S+1,...,2n+4,2n+6\} \) in that order.

Then \( e_f(0) = e_f(1) = k \) where \( k = \frac{2n+6}{2} \)

Hence \( |e_f(0) - e_f(1)| = 0 \)

Hence, the graph \( J(n+1,n) \) is a Near divisor cordial.

THEOREM 2.9:
The shell \( S_n \) is a Near divisor cordial.

Proof:

Let \( V(S_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\} \)

\( E(S_n) = \{v_0v_i, 1 \leq i \leq n-1\} \cup \{v_i v_{i+1}, 1 \leq i \leq n-2\} \)

Fix \( f(v_0) = 1 \)

Label the remaining vertices from \( \{3,4,...,n-1,n+1\} \) in that order.

Then \( e_f(0) = n-2 \), \( e_f(1) = n-1 \)

Hence \( |e_f(0) - e_f(1)| = 1 \)

Hence, \( S_n \) is a Near divisor cordial.

Example 2.10:

\[ S_5 \]

THEOREM 2.11:
The graph \( B_n,m^2 \) is Near divisor cordial.

Proof:

Let \( |V(B_n,m^2)| = n+m+2 \) and \( |E(B_n,m^2)| = 4n+1 \)

Always fix \( f(v_0) = 1 \) and \( f(u_0) = S \), where \( S \) is the largest prime such that \( S \leq n+m+3 \) and \( S \neq n+m+2 \).

Then label the remaining vertices from \( \{2,3,...,n+m+1,n+m+3\} \)

Then \( e_f(0) = n+m \), \( e_f(1) = n+m+1 \)

Hence \( |e_f(0) - e_f(1)| = 1 \)

Hence, The graph \( B_n,m^2 \) is Near divisor cordial.

Example 2.12:

THEOREM 2.13:
The star graph \( K_1,n \) is Near divisor cordial if \( n \) is odd, \( n \geq 5 \).

Proof:

Let \( V(K_1,n) = \{v_1,v_2,v_3,\ldots,v_n\} \) and \( E(K_1,n) = \{v_i v_{i+1}, 1 \leq i \leq n\} \) and \( n=4 \).

Now assign label 2 to the vertex \( v \) and label the remaining vertices \( v_1, v_2, v_3, \ldots, v_n \) by \( 1,3,4,\ldots,\ldots, n-1 \) and \( n+1 \) respectively.

We have, \( e_f(0) = k+1 \), \( e_f(1) = k \), where \( n=2k+1 \)

Hence, \( |e_f(0) - e_f(1)| = 1 \)

Therefore, \( K_1,n \) is Near divisor cordial for \( n \) is odd and \( n \geq 5 \), \( n = 4,6 \)

Conversely,

Suppose \( K_1,n \) is Near divisor cordial.

Suppose \( n \) is even and \( n \geq 8 \)

Let \( n = 2k \), there are \( k+1 \) even numbers and \( k \) odd numbers as labels.

Assigning any odd number to the central vertex as label, then it does not satisfy the condition \( |e_f(0) - e_f(1)| \leq 1 \).

If \( f(v) = 2 \) for a labelling \( f \) in that case also \( e_f(1) = k+1 \) and \( e_f(0) = k-1 \). It can be easily verifies that by assigning any even number > 2 to the central vertex as label, then it does not satisfy the condition \( |e_f(0) - e_f(1)| \leq 1 \).

\( \therefore \) \( K_1,n \) is not Near divisor cordial.

Clearly, \( K_1,n \equiv P_3 \) is not near divisor cordial.

Therefore if \( K_1,n \) is Near divisor cordial then \( n \) should be odd and \( n \geq 3 \) and \( n=4,6 \).

Example 2.14:
THEOREM 2.15:

S(K1,n) the subdivision of the star k 1,n is near divisor cordial

Proof:

Let V(S(K1,n)) = {v, vi, ui : 1 ≤ i ≤ n} and E(S(K1,n)) = {vvi, viui : 1 ≤ i ≤ n}

Define f by:

\[ f(v) = 1 \]
\[ f(vi) = 2i \quad (1 ≤ i ≤ n) \]
\[ f(ui) = 2i+1 \quad (1 ≤ i ≤ n-1) \]
\[ f(un) = 2i+2 \text{ where } i = n \]

Here, e_f(0) = e_f(1) = n

Hence, \[ |e_f(0) - e_f(1)| = 0 \]

Therefore, S(K1,n) is near divisor cordial.

Example 2.16:

THEOREM 2.17:

The complete bipartite graph K2,n is near divisor cordial.

Proof:

Let V(K2,n) = V1UV2

Such that \[ |V(K2,n)| = n+2 \text{ and } |E(K2,n)| = 2n. \]

V1 = {x1, x2} and V2 = {v1, v2, v3, ...., vn}. Now define f, such that f(x1) = 1, f(x2) = 2 and f(x3) = s, where s is the largest prime number such that s ≤ n+4 and s ≠ n+3.

And assign the remaining labels to the vertices y1, y2, y3, ...., yn

Then, e_f(0) = n, e_f(1) = n-1

Hence, \[ |e_f(0) - e_f(1)| = 1 \]

Thus, K2,n is a Near divisor cordial.

Remark 2.21:

For Km,n, m ≥ 4, e_f(0) value increases drastically than e_f(1) and it is true for any Near divisor cordial labelling f except for some particular values of m & n.

THEOREM 2.22:

The graph G = <K1,n(1),K1,n(2)> is Near divisor cordial.

Proof:

Let v1(1), v2(1), ...., vn(1) be the pendant vertices of K1,n(1) and v1(2), v2(2), ...., vn(2) be the pendant vertices of K1,n(2).

Let c1 and c2 be the apex vertices of K1,n(1) and K1,n(2) respectively and they are adjacent to a common vertex w. \[ |V(G)| = 2n+3 \quad |E(G)| = 2n+2. \]

Let f : V(G) → {1, 2, 3, ...., 2n+2, 2n+4}

Now, assign the label 1 to c1 and the largest prime number S such that S ≤ 2n+4. Then assign the remaining vertices y1, y2, y3, ...., yn to be labelled to the remaining vertices of G. Since 1 divides any integer, and S does not divide any integer, then e_f(0) = n+1 and e_f(1) = n+1

Hence, \[ |e_f(0) - e_f(1)| = 0 \]

Hence, the graph G = <K1,n(1),K1,n(2)> is Near divisor cordial.

Example 2.23:
Theorem 2.24:
The graph \( G = < K_{1,n(1)}, K_{1,n(2)}, K_{1,n(3)} > \) is Near divisor cordial

Proof:
Let \( v_1(1), v_2(1), \ldots, v_n(1) \) be the pendent vertices of \( K_{1,n(1)} \) and \( v_1(2), v_2(2), \ldots, v_n(2) \) be the pendent vertices of \( K_{1,n(2)} \) and \( v_1(3), v_2(3), \ldots, v_n(3) \) be the pendent vertices of \( K_{1,n(3)} \).

Let \( c_1 \) and \( c_2 \) and \( c_3 \) be the apex vertices of \( K_{1,n(1)} \) and \( K_{1,n(2)} \) and \( K_{1,n(3)} \) respectively and they are adjacent to a common vertex \( w_1 \) and \( w_2 \) such that \( w_1 \) is adjacent to \( c_1 \) and \( c_2 \) and \( w_2 \) is adjacent to \( c_2 \) and \( c_3 \).

Note that \( G \) has \( 3n+5 \) vertices and \( 3n+4 \) edges.

Case 1: \( n \) is odd
Now assign the label 1 to \( c_1 \), 2 to \( c_2 \) and \( S \) to \( c_3 \) where \( S \) is the largest prime number such that \( S \leq 3n+5 \) (and \( S \neq 3n+4 \)). Then assign the remaining numbers to the pendent vertices, we get,

\[
\text{ef}(1) = \frac{3n+5}{2} \quad \text{and} \quad \text{ef}(0) = \frac{3n+3}{2} \quad \text{(See fig.)}
\]

Then, \( |\text{ef}(0) - \text{ef}(1)| = 1 \).

Case 2: \( n \) is even
Now assign the label 1 to \( c_1 \), \( S \) to \( c_2 \), where \( S \) is the largest prime number such that \( S \leq 3n+5 \) and 2 to \( c_3 \). Then assign the remaining numbers to the pendent vertices in such a way that \( \frac{n+1}{2} \) vertices adjacent to \( C_3 \) is assigned even numbers and remaining \( \frac{n+1}{2} \) vertices adjacent to \( C_3 \) is assigned odd numbers and the remaining labels are assigned to the left over pendent vertices.

We get, \( \text{ef}(1) = \frac{3n+4}{2} = \text{ef}(0) \quad \text{(See fig.)} \)

Then, \( |\text{ef}(0) - \text{ef}(1)| = 1 \).

Hence, The graph \( G = < K_{1,n(1)}, K_{1,n(2)}, K_{1,n(3)} > \) is Near divisor cordial.

Example 2.25: