Manisha Malik et al. International Journal of Recent Research Aspects ISSN: 2349-7688, Vol. 5, Issue 1, March 2018, pp. 479-484

Modeling, Analysis and Simulation of Homogenization Methods for Partial Differential Equations

Manisha Malik

M.Sc. (H.S.) Mathematics, Panjab University
Email: malikmanisha1995@gmail.com

Abstract- **The main interest is on Aysmptotic Analysis of partial differential equations. This is a technique to understand the macroscopic behaviour of a composite medium through its microscopic properties. The technique is commonly used for PDE with highly oscillating coefficients. The idea is to replace a given heterogeneous medium by a fictitious homogeneous one (the `homogenized' material) for numerical computations. The technique is also known as ``Multi scale analysis''. The known and unknown quantities in the study of physical or mechanical processes in a medium with micro structure depend on a small parameter ε. The study of the limit as $ \varepsilon \rightarrow 0 $, is the aim of the mathematical theory of homogenization. The notion of G-convergence, H-convergence, two-scale convergence are some examples of the techniques employed for specific cases. The variational characterization of the technique for problems in calculus of variations is given by Γ-convergence. This paper presents a few examples. Given the complexity of these processes, the key to reliably simulate some relevant classes of such processes involves the construction of appropriate macroscopic (homogenized or effective) models. This is illustrated by studying a one-dimensional model problem of oscillatory diffusion**

Keywords— Partial Differential Equations, Variational Methods, Homogenization Methods Simulation

I. Introduction

Heterogeneous micro-structures on many non-separable scales and high contrast in physical properties of the constituents are key features for the superior behaviour of modern composite and multi-functional materials. However, these features cause major difficulties for their computer simulation. The resolution of all characteristic length scales is prohibitively expensive while the naive disregard of relevant microscopic information leads to questionable results, even on macroscopic scales of interest. Homogenization methods try to remedy this dilemma. They account for the relevant microscopic information in a hierarchical, concurrent and adaptive fashion so that a reliable simulation of multiscale problems eventually becomes feasible in state-of-the-art computing environments. This lecture concerns the design of the related numerical algorithms and, equally important, the mathematics behind them to foresee and assess their reliability and efficiency in engineering and scientific applications. Among the target applications of this lecture is the mechanical analysis of multiphase materials such as composite and multifunctional materials. The manipulation of characteristics and relative volumes of its constituents allows one to equip engineered multiphase materials with some targeted portfolio of physical properties (e.g. light-weight, stiffness, strong electric and magnetic order, energy conversion). The development of novel multifunctional materials for the next-generation of performance-tailored structures requires the topological optimisation of the microstructures and, hence, the understanding how certain material properties (conductivity, permeability, etc.) depend on controllable variables (thermal conductivities of the constituents, relative volumes, particles shapes, coating and size). Transport processes in porous media, e.g. groundwater flow in unsaturated soils [16, 18], share the previous challenges in that the occurring permeabilities and hydraulic conductivities have rapidly changing features due to different types of soil, microscopic inclusions in the bottom or porous subsurface rock formations. Any meaningful numerical simulation of relevant physical effects has to account for these highly heterogeneous fine scale structures in the whole computational domain. If pore scale effects become relevant or if domains spread over kilometers, the computational load easily exceeds computer capacity when standard finite element or finite volume methods are used.

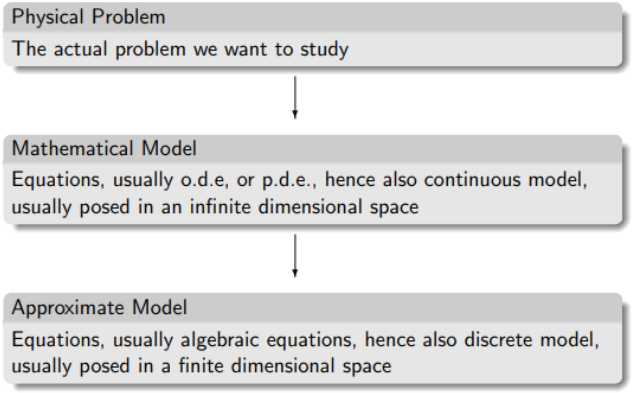

Fig. 1 Modeling

Numerical homogenization methods are techniques for finding numerical solutions of partial differential equations (PDEs) with rapidly oscillating coefficients (multiple scales) [26]. In mathematical analysis, homogenization can be defined as a theory for replacing a PDE with rapidly oscillating coefficients by a PDE with averaged coefficients (an effective PDE), that describes the macroscopic behavior

of the original equation. Numerical techniques that are able to approximate the solution of an effective PDE (often unknown in closed form) and local fluctuation of the oscillatory solution without resolving the full oscillatory equation by direct discretization are coined "numerical homogenization methods" [27]. These methods are also called multiscale methods as they typically combine numerical solvers on different scales.

II. MOTIVATION

The modeling of the physical processes in strongly heterogeneous medium motivates the study of partial differential equations with oscillating coefficients. The basic problem is this: a family of physical processes are assumed to solve Aεuε = fε (1.1) with appropriate initial and/or boundary conditions [28].

Here, Aε is a differential operator, and the parameter ε stands for the period of oscillation. If the knowledge of small scale variation of the heterogeneous medium is only known to statistical extent, the medium is modeled as a random field. We are interested in deriving a homogenized equation that captures the effective properties of the heterogeneous medium, as the computational cost of solving the multi-scaled equation (1.1) is prohibitive [29]. Specifically, we would like to understand the following issues:

EXAMPLE

(1) Does the solution uε converge? If the limiting solution u solve the equation Au = f, (1.2) we call (1.2) a homogenized equation. (2) Which type of convergence do we have in (1)? Is it a L 2 convergence to a deterministic PDE, or is it a weak convergence to a stochastic PDE? (3) What is the rate of convergence? In other words, can we prove kuε − uk ≤ Cεγ (1.3) for some γ > 0. 1.2 Example Let us see an example of the random homogenization problem:

$$\frac{\partial}{\partial t}u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + \frac{1}{\varepsilon^\alpha}V(\frac{x}{\varepsilon})u_\varepsilon(t,x) = 0$$
$$u_\varepsilon(0,x) = u_0(x). \quad (1.4)$$

This equation can be seen as a continuous version of the parabolic Anderson model. The asymptotic behavior of this equation depends on the dimension d. For simplicity, we assume V to be a Gaussian field.

For d = 1 and α = 1/2, uε converges weakly to the stochastic PDE

$$\frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) + \sigma u(t,x) \circ \dot{W} = 0,$$
$$u(0,x) = u_0(x), \quad (1.5)$$

where W˙ denotes spatial white noise,

◦ denotes Stratonovich product, and

$$\sigma^2 := \int_{\mathbb{R}^d} \mathbb{E}\{V(0)V(x)\}dx. \quad (1.6)$$

For d = 2 and ε α := ε| log ε|, or d > 2 and α = 1,

uε converges in L 2 (Ω × R d) to the deterministic PDE

$$\frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) - \rho u(t,x) = 0,$$
$$u_\varepsilon(0,x) = u_0(x), \quad (1.7)$$

where

$$\rho := \begin{cases} c_d\hat{R}(0), & d = 2 \\ \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2}d\xi, & d > 2. \end{cases} \quad (1.8)$$

Here, cd is the volume of unit sphere on (d−1)-dimensional hyperplane. Rˆ(ξ) is the Fourier transform of the covariance function

R(x) = E{V (y)V (x + y)}.

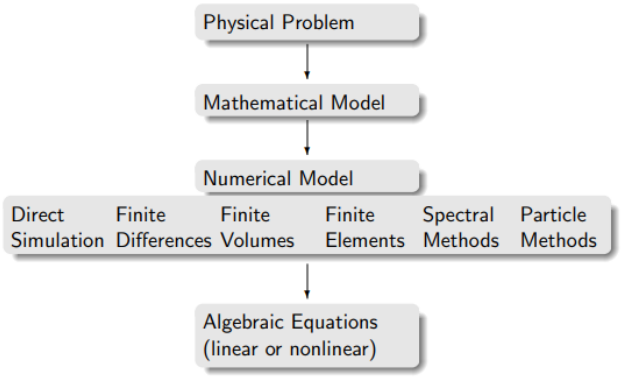

Fig. 2 Modeling and Simulation

Homogenization Consider a general family of PDEs Lε(uε) = f with oscillating coefficients depending on a small parameter ε > 0 with solution uε : Ω −→ R, where Ω is an open subset of R d , 1 ≤ d ≤ 3. The parameter ε emphasizes the multiscale nature of the above family of PDEs, and represents a typical microscopic length scale of a heterogeneity in the system (multiple microscopic length scales could be considered as well) [30]. One can think of the solution as containing low O(1) frequency components and high O(1/ε) frequency components. Solving numerically a given PDE of the above family using classical numerical approximations such as the finite element method (FEM) [31], the finite difference method (FDM) or the finite volume method (FVM), would usually amount in a number of degrees of freedom (DOF) (or unknowns of the discrete system) proportional to O(ε −d), which can be prohibitive for small ε. If the family of solutions converges (in some appropriate sense) to a limit denoted u0 when the size of the heterogeneity ε → 0 and if that limit is the solution of an averaged (homogenized) equation L0(u0) = f, we then have an effective (upscaled, averaged) model that can be treated with a classical method at a cost independent of ε. The rigorous study of these questions is the core of the mathematical homogenization theory [10; 26; 28].

III. NUMERICAL APPROACHES

In most practical situations, the averaged equation described in the previous section is not known in explicit form. Furthermore, even if known, the data of the averaged equation are usually not known explicitly but rely for each x ∈ Ω on yet another PDE. Numerical approaches for homogenization problems were pioneered by Babuˇska [8] and have since then enjoyed considerable developments. In what follows we explain the main ideas of a few numerical homogenization strategies that have been developed in the applied mathematics community. There is also an abundant related literature on multiscale computational methods in the field of material sciences, that share similar ideas as the ones described below (unit cell methods, continuous/discontinuous

computational homogenization methods). The emphasis there is rather on applications (bulk modeling, crack modeling, failure) and we refer to recent reviews for references [27; 21]. Among the computational methods that we will describe, we will focus on techniques based on finite element methods (FEMs), but the main ideas are also applicable to other type of discretizations.

We choose for Lε(uε) = f an elliptic multiscale problem that reads in weak form: Find uε ∈ V (Ω) such that

B(uε, v) = Z Ω a ε∇uε · ∇vdx = (f, v) ∀v ∈ V (Ω), (1)

where (f, v) = R Ω fvdx and V (Ω) is a Sobolev space that we choose to be H1 0 (Ω) (the space of square-integrable functions that vanishes on ∂Ω with square-integrable derivatives).

Here a ε is an oscillating tensor with fast O(1/ε) and slow frequencies. The homogenized problem corresponding to the above equation reads: Find u0 ∈ V (Ω) such that

B0(u0, v) = Z Ω a 0∇u0 · ∇vdx = (f, v) ∀v ∈ V (Ω). (2)

The solution uε can be expected to behave as u0 + εu1, with ku1kL2(Ω) = O(1) but k∇u1kL2(Ω) = O(1/ε). A standard finite element (FE) approximation of (1) consists in 4 a solution uh of (1) in a finite dimensional space spanned by piecewise polynomials on a partition Th of Ω with mesh size h (see below). However, a good approximation of uε by uh (the FE solution) is usually obtained only if h ε in which case the complexity (DOF) scales as O(ε −d). Two main classes of numerical homogenization methods have been developed to address this issue: 1. methods based on a reduced model generated from the original fine scale problem, 2. methods that sample the original fine scale problem on patches to recover effective data of a macroscopic model and use correctors to reconstruct the fine scale solution. Notations In what follows we will consider for simplicity Ω to be both polygonal and convex and we restrict ourselves to simplicial FEs. We consider a family of macroscopic (conformal, shape regular) triangulations TH of Ω = ∪K∈TH K, with elements K of diameter HK and H = maxK∈TH the size of the triangulation (mesh size).

For a macroscopic triangulation, H > ε is allowed.

On a (polygonal) subset D of Ω we also consider a microscopic triangulation

D = ∪T ∈Th T,

with elements T of diameter hT and a meshsize h that satisfies h < ε. We then consider the following FE spaces

VH(Ω) = {vH ∈ V (Ω); vH|K ∈ P1 (K), ∀K ∈ Th}, (3)

Vh(D) = {vh ∈ V (D); vh|T ∈ P1 (T), ∀T ∈ Th}, (4)

where P 1 (K) is the space of piecewise linear polynomials on K (resp. T).

For a cubic domain D = Y we also consider

Wh(D) = {vh ∈ W1 per(D);

vh|T ∈ P1 (T), ∀T ∈ Th}, (5)

where W1 per(D) is a Sobolev space of periodic functions (the closure of smooth periodic functions on D for the H1 norm, where functions differing by a constant are identified). We consider here piecewise linear polynomials and conformal meshes for simplicity but 5 emphasize that the methods described below have been generalized to higher order piecewise polynomial spaces and other types of FEs.

IV. Homogenization of PDEs with Gaussian coefficients

Gaussian fields are unique among all random fields in that the first and second order moments (mean and covariance) completely determines the distribution. Without loss of generality, we assume Gaussian fields to have zero mean hereafter. For Gaussian fields, moments of all orders can be calculated in terms of the second order moments.

Definition 2.6. A Gaussian field is a random field involving only Gaussian distributed random variables.

Theorem 2.7. If (Z1, · · · , Zn) is a zero mean multivariate Gaussian random vector, then.

$$\mathbb{E}[Z_1 \cdots Z_n] = \begin{cases} 0, & n \text{ is odd} \\ \sum \prod \mathbb{E}[Z_i Z_j], & n \text{ is even,} \end{cases}$$

where the notation P Q means summing over all distinct ways of partitioning Z1, · · · , Zn into pairs.

V. Setting of the problem

We consider a model problem of diffusion or conductivity in a periodic medium (for example, an heterogeneous domain obtained by mixing periodically two different phases, one being the matrix and the other the inclusions; see Figure 1.1). To fix ideas, the periodic domain is called Ω (a bounded open set in R N with N ≥ 1 the space dimension), its period ǫ (a positive number which is assumed to be very small in comparison with the size of the domain), and the rescaled unit periodic cell Y = (0, 1)N . The conductivity in Ω is not constant, but varies periodically with period ǫ in each direction. It is a matrix (a second order tensor) A(y), where y = x/ǫ ∈ Y is the fast periodic variable, while x ∈ Ω is the slow variable.

Equivalently, x is also called the macroscopic variable, and y the microscopic variable. Since the component conductors do not need to be isotropic, the matrix A can be any second order tensor that is bounded and positive definite, i.e., there exist two positive constants β ≤ α > 0 such that, for any vector ξ ∈ R N and at any point

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(y)\xi_i\xi_j \leq \beta|\xi|^2. \quad (1.1)$$

At this point, the matrix A is not necessarily symmetric (such is the case when some drift is taken into account in the diffusion process). The matrix A(y) is a periodic function of y, with period Y , and it may be discontinuous in y (to model the discontinuity of conductivities from one phase to the other). Denoting by f(x) the source term (a scalar function defined in Ω), and enforcing a Dirichlet boundary condition (for simplicity), our model problem of conductivity reads

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

A periodic domain. where uǫ(x) is the unknown function, modeling the electrical potential or the temperature. Remark 1.1.1 From a mathematical point of view, problem (1.2) is well posed in the sense that, if the source term f(x) belongs to the space L 2 (Ω) of square integrable functions on Ω, then the Lax-Milgram lemma implies existence and uniqueness of the solution uǫ in the Sobolev space H1 0 (Ω) of functions which belong to L 2 (Ω) along with their first derivatives. Furthermore, the following energy estimate holds kuǫkL2(Ω)

+ k∇uϱkL2(Ω) ≤ C, where the constant C does not depend on ϱ. The domain Ω, with its conductivity A x ϱ , is highly heterogeneous with periodic heterogeneities of lengthscale ϱ. Usually one does not need the full details of the variations of the potential or temperature uϱ, but rather some global of averaged behavior of the domain Ω considered as an homogeneous domain. In other words, an effective or equivalent macroscopic conductivity of Ω is sought. From a numerical point of view, solving equation (1.2) by any method will require too much effort if ϱ is small since the number of elements (or degrees of freedom) for a fixed level of accuracy grows like 1/ϱN . It is thus preferable to average or homogenize the properties of Ω and compute an approximation of uϱ on a coarse mesh.

Averaging the solution of (1.2) and finding the effective properties of the domain Ω is what we call homogenization. There is a difference of methodology between the traditional physical approach of homogenization and the mathematical theory of homogenization. In the mechanical literature, the so-called representative volume element (RVE) method is often used (see [8], or section 1 in [17]). medium, and averaging over it the gradient ∇uϱ and the flux A x ϱ ∇uϱ. Denoting by ξ the average of the gradient and by σ that of the flux, the effective tensor of conductivity A∗ of this sample is defined by the linear relationship σ = A∗ ξ. It turns out that the averaged stored energy A x ϱ ∇uϱ · ∇uϱ is also equal to the effective energy A∗ ξ · ξ. Although this type of definition is very intuitive, it is not clear whether it defines correctly an effective tensor A∗ . In particular, it may depend on the choice of source term f, sample size, or boundary conditions. The mathematical theory of homogenization works completely differently.

Rather than considering a single heterogeneous medium with a fixed lengthscale, the problem is first embedded in a sequence of similar problems for which the lengthscale ϱ, becoming increasingly small, goes to zero. Then, an asymptotic analysis is performed as ϱ tends to zero, and the conductivity tensor of the limit problem is said to be the effective or homogenized conductivity. This seemingly more complex approach has the advantage of defining uniquely the homogenized properties. Further, the approximation made by using effective properties instead of the true microscopic coefficients can be rigorously justified by quantifying the resulting error.

In the case of a periodic medium Ω, this asymptotic analysis of equation (1.2), as the period ϱ goes to zero, is especially simple. The solution uϱ is written as a power series in ϱ uϱ = X +∞ i=0 ϱ iui . The first term u0 of this series will be identified with the solution of the so-called homogenized equation whose effective conductivity A∗ can be exactly computed. It turns out that A∗ is a constant tensor, describing a homogeneous medium, which is independent of f and of the boundary conditions.

Therefore, numerical computations on the homogenized equation do not require a fine mesh since the heterogeneities of size ϱ have been averaged out. This homogenized tensor A∗ is almost never a usual average (arithmetic or harmonic) of A(y). Various estimates will confirm this asymptotic analysis by telling in which sense uϱ is close to u0 as ϱ tends to zero. Remark 1.1.2 From a more theoretical point of view, homogenization can be interpreted as follows. Rather than studying a single problem (1.2) for the physically relevant value of ϱ, we consider a sequence of such problems indexed by the period ϱ, which is now regarded as a small parameter going to zero.

The question is to find the limit of this sequence of problems. The notion of limit problem is defined by considering the convergence of the sequence (uϱ)ϱ>0 of solutions of (1.2): Denoting by u its limit, the limit problem is defined as the problem for which u is a solution. Of course, u will turn out to coincide with u0, the first term in the series defined above, and it is therefore the solution of the homogenized equation. Clearly the mathematical difficulty is to define an adequate topology for this notion of convergence of problems as ϱ goes to zero.

VI. IMPLEMENTATION OF PHYSICAL-STATISTICAL MODEL WITH HOMOGENIZATION

In terms of implementation, one can proceed as described in Section 1.2 where, depending on the type of observations that are made, a suitable statistical data model can be chosen along with an error distribution for the process model and prior for the parameters. Assuming that the data consist of observations of animal abundance (i.e., N(x,t), counts of animals; discussed in more detail in the following section) and we make the appropriate homogenization transformations, we can treat the harmonic mean as an operator using vector notation (i.e., $\bar{\delta} \equiv \bar{\delta}(\delta)$) in the following general hierarchical statistical model:

$$N(x,t) \sim [N(x,t) \mid u_0(x,t)], \quad \forall x,t$$
$$u_0(x,t) \sim [u_0(x,t) \mid f_h(\mathbf{u}_0(t-\Delta t), \bar{\boldsymbol{\delta}}(\boldsymbol{\delta}))], \quad \forall x,t$$
$$\boldsymbol{\delta} \sim [\boldsymbol{\delta}],$$

where the function fh represents the plain diffusion solver as a difference equation. Note that the process stage (2.9) of the hierarchical model could either be stochastic or a degenerate distribution implying no additional process uncertainty beyond that provided through the data model (2.8). Though perhaps possible, it would not be trivial to fit such a model and obtain uncertainty estimates for model parameters using maximum likelihood. Therefore, we describe a Bayesian approach to fit the model that uses an MCMC algorithm as described in recent literature pertaining to physical-statistical modeling (e.g., Wikle and Hooten 2010). The posterior distribution corresponding to the model specified in (1.8)–(1.10) can be expressed as

$$[\{\mathbf{u}_0(t)\}, \boldsymbol{\delta} \mid \{\mathbf{N}(t)\}] \propto \prod_t [\mathbf{N}(t) \mid \mathbf{u}_0(t)] \prod_t [\mathbf{u}_0(t) \mid f(\mathbf{u}_0(t-\Delta t), \boldsymbol{\delta})][\boldsymbol{\delta}].$$

It involves a product over the homogenized process distribution rather than the non-homogenized process. This model is completely non-conjugate, implying that full-conditional distributions cannot be found analytically, thus, a Metropolis–Hastings approach must be used to sample from the full-conditionals sequentially. The portion of an MCMC algorithm where the homogenization technique helps the most is when sampling the diffusion coefficients δ.

VII. RESULT AND DISCUSSION

Statistical models using partial differential equations (PDEs) to describe dynamically evolving natural systems are appearing in the scientific literature with some regularity in

recent years. Often such studies seek to characterize the dynamics of temporal or spatio-temporal phenomena such as invasive species, consumer-resource interactions, community evolution, and resource selection. Specifically, in the spatial setting, data are often available at varying spatial and temporal scales. Additionally, the necessary numerical integration of a PDE may be computationally infeasible over the spatial support of interest. We present an approach to impose computationally advantageous changes of support in statistical implementations of PDE models and demonstrate its utility through simulation using a form of PDE known as "ecological diffusion." We also apply a statistical ecological diffusion model to a data set involving the spread of mountain pine beetle (Dendroctonus ponderosae).

REFERENCES

[1]. Aarts, G., Fieberg, J., and Matthiopoulos, J. (2012), "Comparative Interpretation of Count, Presence–Absence and Point Methods for Species Distribution Models," Methods in Ecology and Evolution, 3, 177–187.

[2]. Alfaro, R., Campbell, R., Vera, P., Hawkes, B., and Shore, T. (2004), "Dendroecological Reconstruction of Mountain Pine Beetle Outbreaks in the Chilcotin Plateau of British Columbia," in Mountain Pine Beetle Symposium: Challenges and Solutions, eds. T. L. Shore, J. E. Brooks, and J. E. Stone, pp. 245–256. Natural Resources Canada, Information Report BC-X-399.

[3]. Aukema, B. H., Carroll, A. L., Zhu, J., Raffa, K. F., Sickley, T. A., and Taylor, S. W. (2006), "Landscape Level Analysis of Mountain Pine Beetle in British Columbia, Canada: Spatiotemporal Development and Spatial Synchrony Within the Present Outbreak," Ecography, 29, 427–441.

[4]. Aukema, B. H., Carroll, A. L., Zheng, Y., Zhu, J., Raffa, K. F., Moore, R. D., Stahl, K., and Taylor, S. W. (2008), "Movement of Outbreak Populations of Mountain Pine Beetle: Influences of Spatiotemporal Patterns and Climate," Ecogeography, 31, 348–358.

[5]. Bentz, B. J. (2006), "Mountain Pine Beetle Population Sampling: Inferences from Lindgren Pheromone Traps and Tree Emergence Cages," Canadian Journal of Forest Research, 36, 351–360.

[6]. Bentz, B. J., Logan, J. A., and Amman, G. D. (1991), "Temperature Dependent Development of the Mountain Pine Beetle (Coleoptera: Scolytidae), and Simulation of Its Phenology," Canadian Entomologist, 123, 1083–1094.

[7]. Bentz, B. J., and Mullins, D. E. (1999), "Ecology of Mountain Pine Beetle (Coleoptera: Scolytidae) Cold Hardening in the Intermountain West," Environmental Entomology, 28, 577–587.

[8]. Bentz, B. J., Powell, J. A., and Logan, J. A. (1996), "Localized Spatial and Temporal Attack Dynamics of the Mountain Pine Beetle (Dendroctonus Ponderosae) in Lodgepole Pine," USDA/FS Research Paper INT-RP494, December, 1996.

[9]. Bentz, B. J., Régnière, J., Fettig, C. J., Hansen, E. M., Hayes, J. L., Hicke, J. A., and Seybold, S. J. (2010), "Climate Change and Bark Beetles of the Western United States and Canada: Direct and Indirect Effects," BioScience, 60, 427–613.

[10]. Berliner, L. M. (1996), "Hierarchical Bayesian Time-Series Models," in Maximum Entropy and Bayesian Methods, Amsterdam: Kluwer Academic, pp. 15–22.

[11]. Biesinger, Z., Powell, J. A., Bentz, B. J., and Logan, J. A. (2000), "Direct and Indirect Parameterization of a Localized Model for the Mountain Pine Beetle Lodgepole Pine System," Ecological Modelling, 129, 273–296.

[12]. Blackard, J. A., Finco, M. V., Helmer, E. H., Holden, G. R., Hoppus, M. L., Jacobs, D. M., Lister, A. J., Moisen, G. G., Nelson, M. D., Riemann, R., Ruefenacht, B., Salajanu, D., Weyermann, D. L., Winterberger, K. C., Brandeis, T. J., Czaplewski, R. L., McRoberts, R. E., Patterson, P. L., and Tycio, R. P. (2008), "Mapping U.S. Forest Biomass Using Nationwide Forest Inventory Data and Moderate Resolution Information," Remote Sensing of Environment, 112, 1658–1677.

[13]. Boone, C. K., Aukema, B. H., Bohlmann, J., Carroll, A. L., and Raffa, K. F. (2011), "Efficacy of Tree Defense Physiology Varies with Bark Beetle Population Density: A Basis for Positive Feedback in Eruptive Species," Canadian Journal of Forest Research, 41, 1174–1188.

[14]. Borden, J. H., Ryker, L. C., Chong, L. J., Pierce, H. D., Johnston, B. D., and Oehlschlage, A. C. (1987), "Response of the Mountain Pine Beetle, Dendroctonus Ponderosae, to Five Semiochemicals in British Columbia Lodgepole Pine Forests," Canadian Journal of Forest Research, 17, 118–128.

[15]. M. B. HOOTEN, M. J. GARLICK, AND J. A. POWELL Cangelosi, A. R., and Hooten, M. B. (2009), "Models for Bounded Systems with Continuous Dynamics," Biometrics, 65, 850–856.

[16]. Crabb, B. A., Powell, J. A., and Bentz, B. J. (2012), "Development and Assessment of 30-m Pine Density Maps for Landscape-Level Modeling of Mountain Pine Beetle Dynamics," Research Paper RMRS-RP-96WWW.

[17]. Fort Collins, CO: U.S. Department Beetle Dynamics. Res. Pap. RMRS-RP-96WWW. Fort

Collins, CO: U.S. Department of Agriculture, Forest Service, Rocky Mountain Research Station, p. 43.

[18]. Cressie, N. A. C., and Wikle, C. K. (2010), Statistics for Spatio-Temporal Data, Hoboken, NJ: Wiley.

[19]. Cressie, N. A. C., Calder, C. A., Clark, J. S., Ver Hoef, J. M., and Wikle, C. K. (2009), "Accounting for Uncertainty in Ecological Analysis: The Strengths and Limitations of Hierarchical Statistical Modeling," Ecological Applications, 19, 553–570.

[20]. Crookston, N. L., Stark, R. W., and Adams, D. L. (1977), "Outbreaks of Mountain Pine Beetle in Northwestern Lodgepole Pine Forests—1945 to 1975," Forest, Wildlife and Range Experiment Station Bulletin No. 22. University of Idaho, Moscow, p. 7.

[21]. Ferreira, M. A. R., and Lee, H. K. H. (2007), Multiscale Modeling, a Bayesian Perspective, New York: Springer. Ferreira, M. A. R., Higdon, D., Lee, H. K. H., and West, M. (2006), "Multiscale and Hidden Resolution Time Series Models," Bayesian Analysis, 1, 947–968.

[22]. Fisher, R. A. (1937), "The Wave of Advance of Advantageous Genes," Annals of Eugenics, 7, 355–369.

[23]. Garlick, M. J., Powell, J. A., and Hooten, M. B. (2011), "Homogenization of Large-Scale Movement Models in Ecology," Bulletin of Mathematical Biology, 73, 2088–2108.

[24]. Gilbert, E., Powell, J. A., Logan, J. A., and Bentz, B. J. (2004), "Comparison of Three Models Predicting Developmental Milestones Given Environmental and Individual Variation," Bulletin of Mathematical Biology, 66, 1821–1850.

[25]. Gotway, C. A., and Young, L. J. (2002), "Combining Incompatible Spatial Data," Journal of the American Statistical Association, 97, 632–648.

[26]. Halsey, R. (1998), Aerial Detection Survey Metadata for the Intermountain Region 4. U.S. Department of Agriculture Forest Service, Forest Health Protection.

[27]. S. Dalal & P. Chahar, Deadlock Resolution Techniques: An Overview in International Journal of Scientific and Research Publications (IJSRP), ISSN 2250-3153, Volume 3, Issue 7, July 2013, pp. 1-6.

[28]. A. Saini, S. Dalal and Dr. Kamal Sharma, A Survey on Outlier Detection in WSN, International Journal of Research Aspects of Engineering and Management ISSN: 2348-6627, Vol. 1, Issue 2, June 2014, pp. 69-72

[29]. U. Rani, S. Dalal, and J. Kumar, "Optimizing performance of fuzzy decision support system with multiple parameter dependency for cloud provider evaluation," Int. J. Eng. Technol., vol. 7, no. 1.2, pp. 61–65, 2018

[30]. Mittal, A and Sharma, KK and Dalal, S, Applying clustering algorithm in case retrieval phase of the case-based reasoning, International Journal of Research Aspects of Engineering and Management, vol. 1, no. 2, pp. 14-16, 2014.

[31]. S. Dalal, G. Tanwar, N. Alhawat, "Designing CBR-BDI agent for implementing supply chain system", System, vol. 3, no. 1, pp. 1288-1292, 2013..

[32]. Hooten, M. B., and Wikle, C. K. (2007), "Shifts in the Spatio-Temporal Growth Dynamics of Shortleaf Pine," Environmental and Ecological Statistics, 14, 207–227.

[33]. Zheng, Y., and Aukema, B. (2010), "Hierarchical Dynamic Modeling of Outbreaks of Mountain Pine Beetle Using Partial Differential Equations," EnvironMetrics, 21, 801–816.

