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Laundau Criteria of Superfluidity of a Mixture of Interacting Bosons and Fermions

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Abstract: **The phenomena of superfluidity was first observed in liquid Helium by Kapitza in 1938. Superfluidity is the friction-less flow of atoms or molecules without dissipation at very low temperatures. There is a critical temperature below which the assembly of interacting particles , bosons , or fermions , or a mixture of bosons and fermions becomes superfluid leading to a phase transition from the so -called normal phase to superfluid phase.Thus there was a need to explain why a mass of atoms could be transported without friction and no dissipation in the superfluid phase . Such a criteria was proposed by Landau according to which the mass transport can take place without dissipation for some critical velocity. To calculate the critical velocity, first the quasi-particle energy expression for an interacting assembly of bosons and fermions is obtained by diagonalizing the model Hamiltonian of an assembly of interacting bosons and fermions . At the minimum value of the momentum in the superfluid state, the critical velocity is calculated .According to Landau's criteria for superfluidity,**

the assembly sustains superfluidity if the velocity of flow is less than the Landau Critical velocity.Themixtures studied are $^{6}_{3}Li$ (fermion)+ $^{7}_{3}Li$ (boson); $^{87}_{37}Rb$ (boson)+ $^{40}_{19}K$ (fermion), $^{23}_{11}Na$ (boson)+ $^{6}_{3}Li$ (fermion), $^{41}_{19}K$ (boson)+ $^{40}_{19}K$ (fermion, $^{23}_{11}Na$ (boson)+ $^{40}_{19}K$ (fermion), $^{7}_{3}Li$ (boson)+ $^{6}_{3}Li$ (fermion)

Key words: Superfluidity,Landau Critical Velocity,Model Hamiltonian,Scattering Length and Quasi-Particle -Energy.

I. INTRODUCTION

The statistical mechanics of identical particles was developed by an Indian Scientist, Satyendra Nath Bose in 1924[1] . The idea of Bose- Einstein -Condensation (BEC)in bosonic gases was predicted by Einstein in 1925[2], and this was based on the ideas of quantum statistics developed by Bose for photons[1]. The state called Bose-Einstein- Condensation (BEC) is obtained when identical bosons at very low temperature ,in the limit $T \to 0K$, share the same wave -function . In BEC, below a critical temperature , T_C , a macroscopic number of particles occupy the lowest energy state, also called the Zero-momentum-state (ZMS). In fact, as the temperature, T , is decreased , the de-Broglie wave length , that varies as $(T)^{-(1/2)}$, increases and at the critical temperature of transition ,it becomes comparable to the inter-particle mean separation . At this stage the wave-function of the particles is very much smeared out such that there is always sufficient overlap of the wave-functions leading to the formation of Bose-Einstein -Condensation (BEC). Hence Bose-Einstein-Condensation is the macroscopic accumulation of non-interacting bosons in the ground state. For a long time the BEC was treated as a theoretical idea since there was no experimental evidence for the existence of many -body -condensed state . The effort to create a BEC experimentally in dilute gases started around 1980[3], although the experiment was not successful due to large rates of inelastic collisions. But after the advances made in laser cooling techniques [4], BEC was achieved in dilute alkali gases experimentally for Rubidium [5], Lithium [6] and sodium [7] vapours. In these experiments , atoms were loaded in a magneto - optical-trap (MOT).Then they are compressed and cooled to very low temperatures (generally micro Kelvin temperatures) using laser cooling methods. Then they are transferred to a magnetic trap where evaporative cooling permits the system to be cooled to nano-Kelvin temperatures. At some critical phase space density, BEC takes place when a macroscopic number of atoms , of the order of 10^6, collectively occupy the lowest energy state (ZMS). It should be pointed out that in liquid helium there exist strong interactions between the particles , whereas in dilute alkali atoms, the interaction between the atoms are relatively weak. In the case of weakly interacting systems, the atoms in the ZMS can be described by a single macroscopic wave function .

Now to create a strongly interacting system , that may be strongly correlated , one has to raise the number density of particles or raise the scattering length or both. Scattering length is raised by using the Feshbach resonance technique [8,9] . A strongly corelated system was, however , obtained by using optical lattices method[10]. By using these methods Bose condensed systems (BEC) and superfluidity were experimentally studied [11].

Experimentally , first observation of BEC, was reached in 1995[7]. Four years later[12], quantum degeneracy was reached in a dilute gas of the fermionic isotope of potassium $\left({}^{40}_{19}K\right)$. Then the process of experimentally cooling mixture of bosonic and fermionic isotopes of lithium was started , and the mixture was cooled to the point of quantum degeneracy [13][14]. After this successful experimental observation leading to quantum degeneracy of the boson -fermion mixture of lithium atoms, a large number of boson -fermion combination of atoms were studied experimentally ,and are given in Table 1

Table 1. Dilute ultra-cold Boson-Fermion gases cooled to quantum degeneracy.

REF	SPECIES
15	${}^{6}Li(f)-{}^{7}Li(b)$
16	${}^{23}Na(b)-{}^{6}Li(f)$
17	${}^{40}K(f)-{}^{87}Rb(b)$
18	${}^{6}Li(f)-{}^{87}Rb(b)$
19	${}^{3}He^{*}(f)-{}^{4}He^{*}(b)$
20	${}^{6}Li(f)-{}^{40}K(f)-{}^{87}Rb(b)$
21	${}^{6}Li(f)-{}^{85,87}Rb(b)$
22	${}^{84,86,88}Sr(b)-{}^{87}Sr(f)$
23	${}^{6}Li(f)-{}^{174}Yb(b)$
24	${}^{170,174}Yb(b)-{}^{173}Yb(f)$
25	${}^{40}K(f)-{}^{41}K(b)-{}^{6}Li(f)$
26	${}^{161}Dy(f)-{}^{162}Dy(b)$
27	${}^{23}Na(b)-{}^{40}K(f)$
28	${}^{6}Li(f)-{}^{133}Cs(b)$
29	${}^{52}Cr(b)-{}^{53}Cr(f)$

In another experiment collective oscillations of superfluid mixture of ultra-cold fermionic and bosonic atoms were investigated by varying boson-fermion scattering length $\left(a_{BF}\right)$[30] and the mixture studied was (${}^{7}_{3}Li-{}^{6}_{3}Li$). In another study a Bose-Einstein condensate of calcium atoms embedded in a degenerate Fermi gas of lithium atoms led to the attractive boson-boson interactions mediated by fermions [31]. Similarly studies were done on Bose-Fermi superfluids [32][33].

By now superfluidity of interacting Bose-Fermi mixture is firmly established. A phenomenological description of a superfluid was proposed by Landau [34] in terms of weakly interacting mixture of excitations such as phonons and rotons. Bogoliubov [35] proposed a field theoretic method to understand Landau's excitation spectrum in which the Hamiltonian of a weakly interacting gas of bosons is diagonalized to obtain quasi-particle excitation spectrum that exhibits a phonon-like spectrum with a linear dispersion at very low momenta. Keeping these developments in mind , the Landau Criteria for superfluidity has been used to obtain an expression for the critical velocity $\left(V_C\right)$ of flow at which a superfluid mixture of interacting bosons and fermions can sustain Values of $\left(V_C\right)$ for different mixtures have been calculated in terms of the boson-fermion scattering length a_{BF} , number density of bosons $\left(n_B\right)$, and the number density of fermions $\left(n_F\right)$.

II. THEORY

According to the Landau criteria for superfluidity , the flow velocity V of the assembly of interacting bosons and fermions has to be lower than some critical velocity V_C $\left(V < V_C\right)$.The value of V_C is to be obtained from the quasi-particle energy E of the assembly when the assembly is in superfluid state. To get the quasi-particle - energy in the superfluid state , first a model Hamiltonian H is to be written for an assembly of interacting bosons and fermions in terms of the creation and annihilation operators for the boson-fermion system [36]. The model Hamiltonian , H , for an assembly of interacting mixture of bosons and fermions is of the form,

$$H = H_B + H_F + H_{BF} \qquad (1)$$

Where

H_B = Hamiltonian for free bosons

H_F = Hamiltonian for free fermions

H_{BF} = Hamiltonian for interacting bosons and fermions.

The model Hamiltonian H can be written in the second -quatized form in terms of the creation and annihilation

$$H = \sum \varepsilon_{kB} a_{kB}^{+} a_{kB} + \sum_{k} \varepsilon_{kF} a_{kF}^{+} a_{kF} \quad (2)$$
$$+ \frac{1}{2} \sum G_{BF} a_{k_1 B}^{+} a_{k_2 F}^{+} a_{k_2' F} a_{k_1' B}$$

Where

ε_{kB} = kinetic energy of the bosons in the state k

ε_{kF} = kinetic energy of the fermions in the state k

G_{BF} = measure of the interaction strength between bosons and fermions.

Assuming momentum conservation and invariance of the two-particle interaction under time reversal $(G_{BF} = G_{-BF})$, and at very low temperatures when the assembly of interacting bosons and fermions is in the condensed phase (superfluid state), the Hamiltonian H can be written as [37],

$$H = \sum_{k} \varepsilon_{kB} a_{kB}^{+} a_{kB} + \sum_{k} \varepsilon_{kF} a_{kF}^{+} a_{kF} + \frac{1}{2} G_{BF}^{0} n_{0B} n_{0F}$$
$$+ \frac{1}{2} G_{BF}^{0} n_{0B} \sum_{k} a_{kF}^{+} a_{kF} + \frac{1}{2} G_{BF}^{0} n_{0F} \sum_{k} a_{kB}^{+} a_{kB}$$
$$- \frac{1}{2} \sum_{k} G_{BF}^{k} a_{kB}^{+} a_{kF} a_{0F}^{+} a_{0B} - \frac{1}{2} \sum_{k} G_{BF}^{-k} a_{0B}^{+} a_{OF} a_{kF}^{+} a_{kB} \quad (3)$$

In Eq.(3), The first term refers to the energy of the free bosons .The second term refers to the energy of the free fermions. The third term refers to the interaction energy of the bosons and fermions in the zero-momentum -state (ZMS). The fourth term corresponds to the interaction energy of the bosons in the ZMS and fermions in the states above $k = 0$ .The fifth term corresponds to the interaction energy of the fermions in the ZMS and the bosons in the states above $k = 0$ .The sixth and the seventh terms correspond to the interaction energy between bosons and fermions in the ZMS and all the levels above $k = 0$.

Now Eq.(3) gives Hamiltonian that describes an assembly of interacting bosons and fermions .It is written in terms of creation and annihilation operators of pure bosons and pure fermions . Whereas we are now to deal with an assembly of interacting bosons and fermions.

To diagonalize the Hamiltonian in Eq.(3), the old operators $(a's)$ are written in terms of some new operators $(\alpha's)$ [38]. Since in an assembly of interacting bosons and fermions , the entity will be a boson-fermion system, terms of the type $a_{kF} a_{kB}^{+}$ and $a_{kB} a_{kF}^{+}$ will operators, a_{kB}^{+} (boson creation operator), a_{kF}^{+} (fermion creation operator) , a_{kB} (boson annihilation operator) , a_{kF} (fermion annihilation operator), such that

appear in the Hamiltonian . Thus a Canonical transformation for a mixture of bosons and fermions was derived to diagonalize the Hamiltonian H [36].After lengthy calculations [37] and substituting the values for ε_{kB} and ε_{kF} [37],the Hamiltonian is diagonalized from which the quasi-particle -energy E in the lowest state (ZMS) or superfluid state is given by [39],

$$E = \frac{1}{2} n_{0B} n_{0F} \left[\frac{4\pi a_{BFO} \hbar^2}{m_{BF}} n_B n_F \right] + \frac{n_B \hbar^2}{2 m_B l^2}$$
$$+ \frac{2\pi a_{B0} \hbar^2 n_B^2}{m_B} + \frac{1}{2} n_{0F} \left[\frac{4\pi a_{BF0} \hbar^2}{m_{BF}} n_B n_F \right] \quad (4)$$
$$+ \frac{n_F \hbar^2}{2 m_F l^2} + \frac{1}{2} n_{0B} \left[\frac{4\pi a_{BF0} \hbar^2}{m_{BF}} n_B n_F \right]$$

In Eq.(4) , we have

n_B = number density of bosons in the mixture;

n_F = number density of fermions in the mixture,

n_{0B} = number density of bosons in the ZMS,

n_{0F} = number density of fermions in the ZMS (below the Fermi surface)

a_{B0} = S-wave scattering length for boson-boson interaction.

a_{BF0} = S-wave scattering length of the colliding boson-fermion pair for boson-fermion interaction.

m_{BF} = reduced mass of an interacting boson-fermion pair.

In Eq.(4), the first term refers to a homogeneous gas of interacting bosons and fermions. The second term is due to the trapping potential for the bosons; the third term is due to the boson -boson interaction Via the S-wave scattering ; the fourth term means that, an interacting system of bosons and fermion in the ZMS is surrounded by isolated fermions (n_{0F}) and the interaction between the bosons and fermions is via the S-wave scattering length a_{BFO} . The fifth term is due to the trapping potential for the fermions .The sixth term represents an interacting system of bosons and fermions in the ZMS, and this system is surrounded by isolated bosons (n_{0B}) , and the interaction between the bosons and fermions is via the S-wave scattering length a_{BF0} . Hence the existence of the term four will demand that $n_F > n_B$, and existence of term six will demand that $n_B > n_F$. The term four can mean demixing of the interacting boson-fermion system from the isolated fermions in the ZMS;and the term six will similarly mean the demixing of the interacting boson-fermion system from the isolated bosons in the ZMS.

Assuming that $n_F = n_B = n$, and that these values are fixed , Eq.(4) can be written as,

$$E = \frac{1}{2}n^2\left(\frac{4\pi a_{BF0}\hbar^2}{m_{BF}}n^2\right) + \frac{n\hbar^2}{2m_B l^2} + \frac{2\pi a_{B0}\hbar^2 n^2}{m_B} + \frac{n\hbar^2}{2m_F l^2} + \frac{1}{2}n\left(\frac{4\pi a_{BF0}\hbar^2}{m_{BF}}n^2\right) \quad (5)$$

However , Eq.(4) should be used to calculate E since the number density of bosons (n_B) and the number density of fermions (n_F)will not be the same. Eq.s(4) and (5) are polynomials in the number density values for bosons and fermions ,and the largest term is the first term. Hence we can write E as , [37,39]

$$E = \frac{1}{2}n_{0B}n_{0F}\left[\frac{4\pi a_{BF0}\hbar^2}{m_{BF}}n_B n_F\right] \quad (6)$$

Rather Eq.(6) can be re-written as, approximately,

$$E = \frac{1}{2}\left[\frac{4\pi a_{BF0}\hbar^2}{m_{BF}}\right]n_{0B}^2 n_{0F}^2 \quad (7)$$

The quasi-particle energy E is the energy of the assembly in the superfluid state.

To determine the Landau critical Velocity, V_C , below which the superfluidity is a stable phenomena is determined by writing an equation such that ,

$$E = pV_C = m_{BF}V_C.V_C = m_{BF}V_C^2 \quad (8)$$

Or $$V_C = \left(\frac{E}{m_{BF}}\right)^{1/2} \quad (9)$$

Eq.(9) is used for different combinations of bosons and fermions gases to calculate the value of V_C . It is clear that the magnitude of V_C depends on the particle number density of the bosons and fermions constituting the interacting mixture ,the reduced mass m_{BF} and the scattering length a_{BF0} in the superfluid state.Keeping n_B, n_F and m_{BF} fixed , the value of a_{BF0} can be changed by Feshbach resonance method . Hence V_C can be changed by changing these parameters.

In fact ,Eqs.(6) and (7) are very similar to the Bose-Fermi interaction energy expression obtained via the Bose - Fermi Hubbard model [40][41] in which the Bose-Fermi interaction energy U_{BF} is written as ,

$$U_{BF} = \frac{2\pi\hbar^2 a_{BF}}{m_{BF}}\int d^3r|\omega_B(r)|^2|\omega_F(r)|^2 \quad (10)$$

Such that the quasi-particle energy can be written as,

$$E = U_{BF}\sum n_B n_F \quad (11)$$

Where

n_B = bosonic number density

n_F = fermionic number density

m_{BF} = reduced mass of interacting boson and fermion pair

a_{BF} = Inter species (boson -fermion) S-wave scattering length

$\omega_B(r)$ = bosonic Wannier function at the position r .

$\omega_F(r)$ = fermionic Wannier function at the position r .

Eq.(11) is the same as Eq.(7) which has been derived by diagonalizing the model Hamiltonian given in Eq.(2). [36,37,39]

In general , the kinetic energies ε_{kB} and ε_{kF} are neglected and this is what is displayed in Eq.(6) and (7). When the shape of the Wannier functions for bosons and fermions is identical (for $^{87}_{37}Rb$ and $^{40}_{19}K$), there will be overlap of the wave functions, but the tunneling rate of the bosons is slower compared to the fermions by a factor of $\frac{m_F}{m_B}$ since $m_B > m_F (m_F v_F = m_B v_B$, Thus $v_B = \frac{m_F}{m_B}v_F)$

In an experiment [41] a mixture of bosonic $^{87}_{37}Rb$ and fermionic $^{40}_{19}K$ atoms were loaded in a three dimensional optical lattice , and the quantum degenerate mixture was composed of $N_B = 4\times10^5$ rubidium atoms, and $N_F = 3\times10^5$ potassium atoms. Using Feshbach resonance method, the inter species scattering length a_{BF} was varied between $-170a_0$ and $+800a_0$ (a_0 = Bohr radius), $-200a_0 < a_{BF} < 400a_0$. The behaviour of condensate fraction was studied at $a_{BF} = -295a_0$ and $a_{BF} = +235a_0$ at $N_F = 2\times10^5$ (by reducing the fermion number). Variation of condensate fraction with lattice depth was also studied ,and it was found that the condensate fraction decays to zero when the lattice depth is decreased . It seems no attempt was made to study the Landau velocity for such mixtures . However , Landau velocity has been calculated in this manuscript by using the above values of a_{BF} and N_B and N_F

Another experimental observation on a mixture of ${}^{7}_{3}Li$ (boson) and ${}^{6}_{3}Li$ (fermion) was done in which the background scattering length a_{BF} for interaction between boson-fermion super fluids was, $a_{BF} = 41a_0$ [42]. This is somewhat smaller than intra - isotope scattering length, but comparable to the Van der Waal's radius for the ${}^{7}_{3}Li - {}^{7}_{3}Li$ and ${}^{6}_{3}Li - {}^{6}_{3}Li$ collision ($R_{vdw} \cong 31a_0$). In this experiment $N_B = 2\times10^5$ and $N_F = 1.5\times10^6$ .In another experiment [43] Feshbach resonances were observed for inter-species ${}^{133}_{55}Cs + {}^{6}_{3}Li$.

III. CALCULATIONS

Table 2. Calculated values of quasi-particle -energy of interacting bosons and fermions mixture.

BOSON - FERMION MIXTURE	n_{0B}	n_{0F}	SCATTERING LENGTH a_{BF} $\times10^{-8} cm$	REDUCED MASS m_{BF} IN g	QUASI PARTICLE ENERGY E IN ergs
${}^{87}_{37}Rb + {}^{40}_{19}K$	10^6	7.5×10^5	150	4.5452×10^{-23}	$1.237414229\times10^{-13}$
			162		1.336407×10^{-13}
	2×10^5	3×10^4	300		2.474828×10^{-13}
			-209		1.7241305×10^{-13}
	4×10^5	2×10^5	-156		1.46484×10^{-15}
			124		1.166579×10^{-15}
${}^{7}_{3}Li + {}^{6}_{3}Li$	4×10^4	3.5×10^5	21.59	5.37703×10^{-24}	5.245906×10^{-17}
			21.69		5.27095×10^{-17}
${}^{23}_{11}Na + {}^{6}_{3}Li$	10^5	10^6	-145	7.9167×10^{-24}	1.2208945×10^{-14}
${}^{41}_{19}K - {}^{40}_{19}K$	10^5	10^4	51.33	3.35994×10^{-23}	1.00933×10^{-19}
${}^{23}_{11}Na - {}^{40}_{19}K$	10^5	10^4	304.277	2.42406×10^{-23}	8.3671×10^{-19}

Table 3. Calculation of Landau's Critical velocity for a Superfluid Mixture of interacting Bosons and Fermions

Boson-Fermion mixture	Energy E (J)	Reduced mass m_{BF} (kg)	Landau Critical Velocity m/s
${}^{87}_{37}Rb + {}^{40}_{19}K$	$1.237414229\times10^{-20}$	4.5452×10^{-26}	521.7
	1.336407×10^{-20}		542.2
	2.474828×10^{-20}		737.8
	1.7241305×10^{-20}		615.9
	1.46484×10^{-22}		56.8
	1.166579×10^{-22}		50.7
${}^{7}_{3}Li + {}^{6}_{3}Li$	5.245906×10^{-24}	5.37703×10^{-27}	31.23
	5.27095×10^{-24}		30.31
${}^{23}_{11}Na + {}^{6}_{3}Li$	1.2208945×10^{-21}	7.9167×10^{-24}	12.42
${}^{41}_{19}K - {}^{40}_{19}K$	1.00933×10^{-26}	3.35994×10^{-26}	0.548
${}^{23}_{11}Na - {}^{40}_{19}K$	8.3671×10^{-26}	2.42406×10^{-26}	1.857

IV. CONCLUSIONS

In this manuscript ,the objective was to calculate the Landau Critical velocity (V_C) for an interacting mixture of bosons and fermions in the superfluid state .For this purpose , first the quasi-particle energy of the mixture is calculated using Eq.(7), and this is tabulated in Table 2 for different mixtures.The Landau Critical velocity (V_C) is calculated from Eq.(9), and its values for different mixtures are tabulated in Table3.

The values of E for different mixtures and different a_{BF} values show that E varies from mixture to mixture depending upon how a_{BF} and the number of bosons and fermions change. The quasi-particle energy, E, varies between $10^{-13}erg$ to $0.5\times10^{-16}erg$, with the exception of the mixtures ${}^{41}_{19}K-{}^{40}_{19}K$ and ${}^{23}_{11}Na-{}^{41}_{19}K$, and this is due to the fact that for these mixtures the values of n_{0B} and n_{0F} are comparatively smaller when compared to other mixtures.The quasi-particle energy of ${}^{4}_{2}He$ superfluid at $1.0K$ is $1.38\times10^{-16}erg$ [44] and the Vander Waal energy between two helium atoms at optimum separation is $\simeq 10^{-16}erg$ [38]. Hence this range of values for E in the superfluid state for the mixture of interacting bosons and fermions is comparable and acceptable.

The value of Landau Velocity (V_C) varies from mixture to mixture , and this is natural since V_C depends on the values of the parameters a_{BF} , n_{0B} and n_{0F} ,and these values are different for different mixtures. The value of V_C for superfluid helium is lower than the velocity of sound $(238m/s)$;its value is $60m/s$ .Table 3 shows that V_C varies from mixture to mixture, and there is a large variation in V_C .The value of V_C is large for mixtures with large values of a_{BF} and n_{0B} and n_{0F} .It is natural that the values of V_C should vary with a_{BF} since the value of V_C for ideal superfluid helium $(238m/s)$ and interacting superfluid helium (60m/s) are very different. Since the stability of the superfluid state depends on the value of V_C ,it is clear from Table 3 that for different mixtures, it is the physical parameters like a_{BF} , n_{0B} and n_{0F} , that will determine the stability of the superfluid state.Even in a given mixture, V_C can be changed by experimentally changing the values or magnitude of the scattering length a_{BF} by Feshbach resonance method. Even the depth of the trapping potential, which is not taken into account in the calculations on E and V_C in this manuscript, could affect the stability of superfluid state and the magnitude of V_C .[41]

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